Appendix-I

Formulation of Dynamic Stiffness Matrix

The expression for n^{th} mode shape (undamped) for vertical vibration of the r^{th} segment of the bridge deck is given by

$$\phi_n(x_r) = A_{nr}\cos\beta_{nr}x_r + B_{nr}\sin\beta nrx_r + C_{nr}\cosh\gamma_{nr}x_r + D_{nr}\sinh\gamma_{nr}x_r \tag{36}$$

where A_{nr} , B_{nr} , C_{nr} and D_{nr} are the integration constants expressed in terms of n^{th} natural frequency of vertical vibration ω_{bn} and

$$\beta_{nr} = \sqrt{\frac{N_r(Z_{nr}+1)}{2E_dI_r}} \;;\; \gamma_{nr} = \sqrt{\frac{N_r(Z_{nr}-1)}{2E_dI_r}} \quad \text{and} \quad Z_{nr} = \sqrt{1+\frac{4E_dI_r\bar{W}_r/g\omega_{bn}^{\;2}}{N_r^{\;2}}}$$

in which E_d and I_r are the modulus of elasticity of the bridge deck and vertical moment of inertia of the beam segment r of the deck respectively; $\bar{W}_r$,g, N_r , N_b, C_r are the dead load per unit length of the beam segment, the acceleration due to gravity, the axial force given to the beam segment r due to cables, number of beam segments and damping of the r^{th} segment of the bridge deck respectively.

Depending on the number of dynamic degrees of freedom there can be four types of dynamic stiffness matrices (Fig. 14).

(a) Moments present at both ends:

Using Eq.36 and expressions for the end displacements $\{X\}_r$ and end forces $\{F\}_r$ at the two ends of any beam segment r, both the end displacements and end forces can be expressed in terms of integration constants as

$$\{X\}_r = [A]_r \{C\}_r \tag{37}$$

and

$$\{F\}_r = [B]_r \{C\}_r \tag{38}$$

where $[A]_r$, $[B]_r$ are 4x4 matrices and $\{C\}_r$ is the vector containing the constants A_{nr}, B_{nr} etc. Expressing $\{C\}_r$ in terms of $\{X\}_r$ from Eq.(37) and substitution of this into Eq.(38) follows

$$\{F\}_r = [B]_r[A]_r^{-1}\{X\}_r$$
$$= [K]_r\{X\}_r$$

Also, from Eq.(37), $\{C\}_r$ may be written as

$$\{C\}_r = [A]_r^{-1}\{X\}_r$$
$$= [R]_r\{X\}_r$$

The elements of the 4 x 4 matrices $[K]_r$ (symmetric) and $[R]_r$ may be expressed as follows:

$k_{11} = G(\sin\lambda\cosh\lambda + \cos\lambda\sinh\lambda)$,

$k_{12} = -G\sin\lambda\sinh\lambda/\beta$, $k_{13} = -G(\sin\lambda + \sinh\lambda)$,

$k_{14} = G(\cos\lambda\cosh\lambda)/\beta$,

$k_{22} = G(\sin\lambda\cosh\lambda - \cos\lambda\sinh\lambda)/\beta^2$, $k_{23} = -k_{14}$,

$k_{24} = -G(\sin\lambda - \sinh\lambda)/\beta^2$, $k_{33} = k_{11}$, $k_{34} = -k_{12}$,

$k_{44} = k_{22}$

where $\beta = \beta_{nr}$, $\lambda = \beta\, l_r$, $G = E\, I_r\, \beta^3/(1-\cos\lambda \cosh\lambda)$, and

$r_{11} = -H(1 + \sin\lambda \sinh \lambda - \cos\lambda \cosh \lambda)$,

$r_{12} = H(\sin\lambda \cosh \lambda - \cos\lambda \sinh \lambda) / \beta$,

$r_{13} = -H(\cos\lambda - \cosh \lambda)$,

$r_{14} = -H(\sin\lambda - \sinh \lambda)/\beta$,

$r_{21} = H(\sin\lambda \cosh \lambda + \cos\lambda \sinh \lambda)$,

$r_{22} = H(1-\sin\lambda \sinh \lambda - \cos\lambda \cosh \lambda)/\beta$,

$r_{23} = -H(\sin\lambda + \sinh \lambda)$,

$r_{24} = -r_{13}/\beta$, $r_{31} = -r_{22}\, \beta$, $r_{32} = -r_{12}$, $r_{33} = -r_{13}$, $r_{34} = -r_{14}$,

$r_{41} = -r_{21}$, $r_{42} = -r_{11}/\beta$, $r_{43} = -r_{23}$, $r_{44} = r_{13}/\beta$,

where $H = 1/2(1 - \cos\lambda \cosh \lambda)$.

(b) Moment zero at the right end:

Eliminating the constant D_{nr} (see Eq. (36)) using the zero moment condition, and following the procedure as in case (a), the elements of the(3 x3) $[K]_r$ matrix and (4 x 3) $[R]_r$ matrix are obtained as:

$k_{11} = 2\, S \cos\lambda \cosh\lambda$,

$k_{12} = -S(\cos\lambda \sinh\lambda + \sin\lambda \cosh\lambda)/\beta$,

$k_{13} = -S(\cos\lambda + \cosh\lambda)$, $k_{22} = 2\, S \sin\lambda \sinh\lambda/\beta^2$,

$k_{23} = S (\sin\lambda + \sinh\lambda)/\beta$, $k_{33} = S(1+ \cos\lambda \cosh\lambda)$

where $S = E\, I_r\, \beta^3/ (\sin\lambda \cosh\lambda - \cos\lambda \sinh\lambda)$, and

$r_{11} = -T \sin\lambda \cosh\lambda$, $r_{12} = T \sin\lambda \sinh\lambda/\beta$,

$r_{13} = T(\sin\lambda + \sinh\lambda)/2$, $r_{21} = T\cos\lambda \cosh\lambda$,

$r_{22} = -T \cos\lambda \sinh\lambda/\beta$, $r_{23} = -T(\cos\lambda + \cosh\lambda)/2$,

$r_{31} = T \cos\lambda \sinh\lambda$, $r_{32} = -r_{12}$, $r_{33} = -r_{13}$,

$r_{41} = (r_{11} \cos\lambda + r_{21} \sin\lambda - r_{31} \cosh\lambda)/ \sinh\lambda$,

$r_{42} = (r_{12} \cos\lambda + r_{22} \sin\lambda - r_{32} \cosh\lambda)/ \sinh\lambda$,

$r_{43} = (r_{13} \cos\lambda + r_{23} \sin\lambda - r_{33} \cosh\lambda)/ \sinh\lambda$

where $T=1/ (\sin\lambda \cosh\lambda - \cos\lambda \sinh\lambda)$.

(c) Moment zero at the left end:

Similarly, eliminating C_{nr} (see Eq.(36)) using zero moment condition, the elements of the two matrices are expressed as

$k_{11} = S(1 + \cos\lambda \cosh\lambda)$, $k_{12} = -S(\cos\lambda + \cosh\lambda)$,

$k_{13} = -S(\sin\lambda + \sinh\lambda)/\beta$, $k_{22} = 2\, S \cos\lambda \cosh\lambda$,

$k_{23} = S(\cos\lambda \sinh\lambda + \sin\lambda \cosh\lambda)/\beta$,

$k_{33} = 2\, S \sin\lambda \sinh\lambda/\beta^2$, and

$r_{11} = -1/2$, $r_{12} = 0$, $r_{13} = 0$,

$r_{21} = T(1 + \cos\lambda \cosh\lambda + \sin\lambda \sinh\lambda)/2$,

$r_{22} = -T \cosh\lambda$, $r_{23} = -T \sinh\lambda/\beta$,

$r_{31} = -1/2$, $r_{32} = 0$, $r_{33} = 0$,

$r_{41} = T(-1 + \sin\lambda \sinh\lambda - \cos\lambda \cosh\lambda)/2$,

$r_{42} = T \cos\lambda$, $r_{43} = T \sin\lambda/\beta$.

(d) Moment zero at both ends:

Eliminating both C_{nr} and D_{nr} (see Eq.(36)) from the zero moment conditions at the ends, the elements of (2 x 2) $[K]_r$ matrix and (4 x 2) $[R]_r$ matrix are derived as

$k_{11} = Q(\cos\lambda \sinh\lambda - \sin\lambda \sinh\lambda)$,

$k_{12} = Q(\sin\lambda - \sinh\lambda)$,

$k_{22} = k_{11}$

where $Q = E I_r \beta^3/ (2 \sin\lambda \sinh\lambda)$, and

$r_{11} = -1/2$, $r_{12} = 0$, $r_{21} = ½ \cot\lambda$, $r_{22} = -1/2 \operatorname{cosec}\lambda$,

$r_{31} = r_{11}$, $r_{32} = r_{12}$, $r_{41} = ½ \coth\lambda$, $r_{42} = -1/2 \operatorname{cosech}\lambda$.