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A Wavelet Based Approach for Estimating the Steady State Response of Nonlinear Systems

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Summary

A wavelet based equivalent linearization method is presented for estimating the steady state response of systems with cubic nonlinearity, excited by non-stationary processes. Harmonic wavelets are used to decompose the excitation into components with non-overlapping frequency components. The original nonlinear system is then decomposed into linear time varying subsystems, each responding only to a particular frequency band of the excitation process. Response for each subsystem is calculated using the equivalent stiffness values obtained form the proposed algorithm. The total response of the original system is then found by adding up the responses from individual subsystems. Finally, records of the Kocaeli, Turkey earthquake (8/17/1999) are used to demonstrate the applicability of the proposed method to a lightly damped oscillator with cubic nonlinearity in stiffness. It has been concluded that the numerical approach developed in this paper gives a reliable estimate of the steady state response of nonlinear systems, excited by non-stationary processes.

Introduction

An inherent difficulty in dealing with nonlinear vibration problems is that the superposition principle is not applicable. The fact that most linear analysis techniques are based on the idea of finding a general solution by superimposing particular solutions, has led many researchers to find alternative formulations for nonlinear analysis (Nigam 1983, Roberts and Spanos 1986, Lin, Y.K. and G.Q. Cai, 1995).

While Fourier transform has been an indispensable tool in signal processing, it is now recognized that by breaking a signal into a series of trigonometric basis functions, time varying features cannot be captured. The fact that non-stationary features characterize many processes of interest has led to the development of alternative transforms that rely on bases with compact support. As Fourier basis functions are localized in frequency but not in time, one alternative was introduced by Gabor to localize the Fourier Transform through the short time Fourier transform. However, the constraints of the Heisenberg uncertainty principle prompt an alternative approach to time-frequency analysis, featuring basis functions that have compact support in both frequency and time to yield a multi resolution analysis called the wavelet transform (Carmona et. al. 1998, Mallat 1989a, Mallat 1989b).

This paper is organized as follows. To illustrate the basic ideas involved, next section is devoted to the discussion of wavelet theory, where Newland's harmonic wavelets are introduced. Later, a wavelet based numerical approach for determining nonlinear system response was developed and the proposed procedure is applied to the response estimation problem. Finally, concluding remarks are made, and other possible applications are discussed.

Wavelet Transform

The wavelet transform is a linear transform, which decomposes a signal x(t) via basis functions that are simply dilations and translations of the mother wavelet ψ(t) through the convolution,

$$W_{\psi}x(s,b) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} x(t)\psi^{*}(\frac{t-b}{s})dt \qquad (1)$$

where $\psi^{*}(t)$ is the complex conjugate of the wavelet function ψ(t), s is the dilation parameter and b is the location parameter of the wavelet. The wavelet coefficients $W_{\psi}x(s,b)$ represent the measure of similarity between the scaled and translated wavelet at and the signal at scale s around time t=b.

The harmonic wavelets were introduced by Newland, in an attempt to create a wavelet basis, equivalent to dilation wavelets with infinite number of coefficients that would occupy a narrow band region in frequency domain (Newland, 1993). Specifically, the mother wavelet of the harmonic transform is defined in the frequency domain as,

$$\begin{aligned} W_{m,n,r}(\omega) &= \frac{1}{(n-m)2\pi} e^{\frac{-i\omega r}{n-m}}, \qquad m2\pi \le \omega \le n2\pi \\ W_{m,n,r}(\omega) &= 0, \quad else, \end{aligned} \qquad (2)$$

where m and n are any positive real numbers. This is the expression for a harmonic wavelet centered at time t=r/(n-m) and frequency (m+n)π with bandwidth (n-m)/(2π).

Wavelets-based Equivalent Linearization Procedure

The most commonly used approach for finding approximate answers to nonlinear dynamic problems is to replace the nonlinear system with a linear system. For a system with nonlinearity in stiffness, one replaces,

$$\ddot{x} + \beta\,\dot{x} + k\,x + h(x) = g(t) \qquad (3)$$

with

$$\ddot{x} + \beta\,\dot{x} + k\,x + k_{nl}x = g(t) \qquad (4)$$

where k_{nl} is the additional stiffness introduced by the nonlinearity. On minimizing the mean square of the error over one cycle of the response, k_{nl} is obtained as,

$$k_{nl} = \frac{\oint h(x)\, x\, dt}{\oint x^2 dt}. \quad (5)$$

From the linear vibration theory, for an excitation described as a sum of sinusoids,

$$g(t) = F_1\ Sin(\omega_1 t) + F_2\ Sin(\omega_2 t) + \ldots + F_N\ Sin(\omega_N t) \quad (6)$$

the steady state response can be written as,

$$x(t) = X_1\ Sin(\omega_1 t + \phi_1) + X_2\ Sin(\omega_2 t + \phi_2) + \ldots + X_N\ Sin(\omega_N t + \phi_N) \quad (7)$$

where,

$$X_i^{\,2} = \frac{F_i^2}{(k - \omega_i^{\,2})^2 + (\beta\omega_i)^2} \quad and \quad \phi_i = \tan^{-1}\frac{\beta\omega_i}{k - \omega_i^2} \quad (8)$$

Substituting x(t) from Eq(7) in Eq(5), and assuming that the stiffness nonlinearity is given by,

$$h(x) = k_{nl}\, x = \lambda\, k\, x^3, \quad (9)$$

the additional stiffness due to the nonlinearity can be found for each frequency of excitation as;

$$\begin{aligned}
k_{nl.1} &= \frac{3}{4}\lambda\, k\, X_1^{\,2}\left[1 + 2\left(\frac{X_2}{X_1}\right)^2 + 2\left(\frac{X_3}{X_1}\right)^2 + \ldots + 2\left(\frac{X_N}{X_1}\right)^2\right] \\
k_{nl.2} &= \frac{3}{4}\lambda\, k\, X_2^{\,2}\left[2\left(\frac{X_1}{X_2}\right)^2 + 1 + 2\left(\frac{X_3}{X_2}\right)^2 + \ldots + 2\left(\frac{X_N}{X_2}\right)^2\right] \\
&\ \vdots \\
&\ \vdots \\
k_{nl.N} &= \frac{3}{4}\lambda\, k\, X_N^{\,2}\left[2\left(\frac{X_1}{X_N}\right)^2 + 2\left(\frac{X_2}{X_N}\right)^2 + 2\left(\frac{X_3}{X_N}\right)^2 + \ldots + 1\right].
\end{aligned} \quad (10)$$

The solution of this system of equations requires an iterative scheme. Starting with

$$k_{nl.i} = 0 \quad and \quad X_i = \frac{F_i}{\sqrt{(-\omega_i^2 + k + k_{nl.i})^2 + (\beta\, \omega_i)^2}} \quad for\ i = 1:N \quad (11)$$

the updated k_{nl} values are calculated using Eq(10). It was observed that the solution converges quickly, usually in a few steps. Total response can then be calculated by summing up all the response components as,

$$x(t) = \sum_{i=1}^{N} x_i(t). \tag{12}$$

As an example, this procedure is applied to the system defined as,

$$\ddot{x} + \beta\,\dot{x} + k\,x + \lambda\,k\,x^3 = g(t) \tag{13}$$

with $\beta=0.5$, $k=16\pi^2$, $\lambda=1$ and g(t)=15 sin(4t)+10 sin(8t)+ 5 sin(12t)+ 5 sin(16t) and the excitation and response processes are shown in (Figure 1). It is seen that the procedure yields a very good approximation of the true response for this problem.

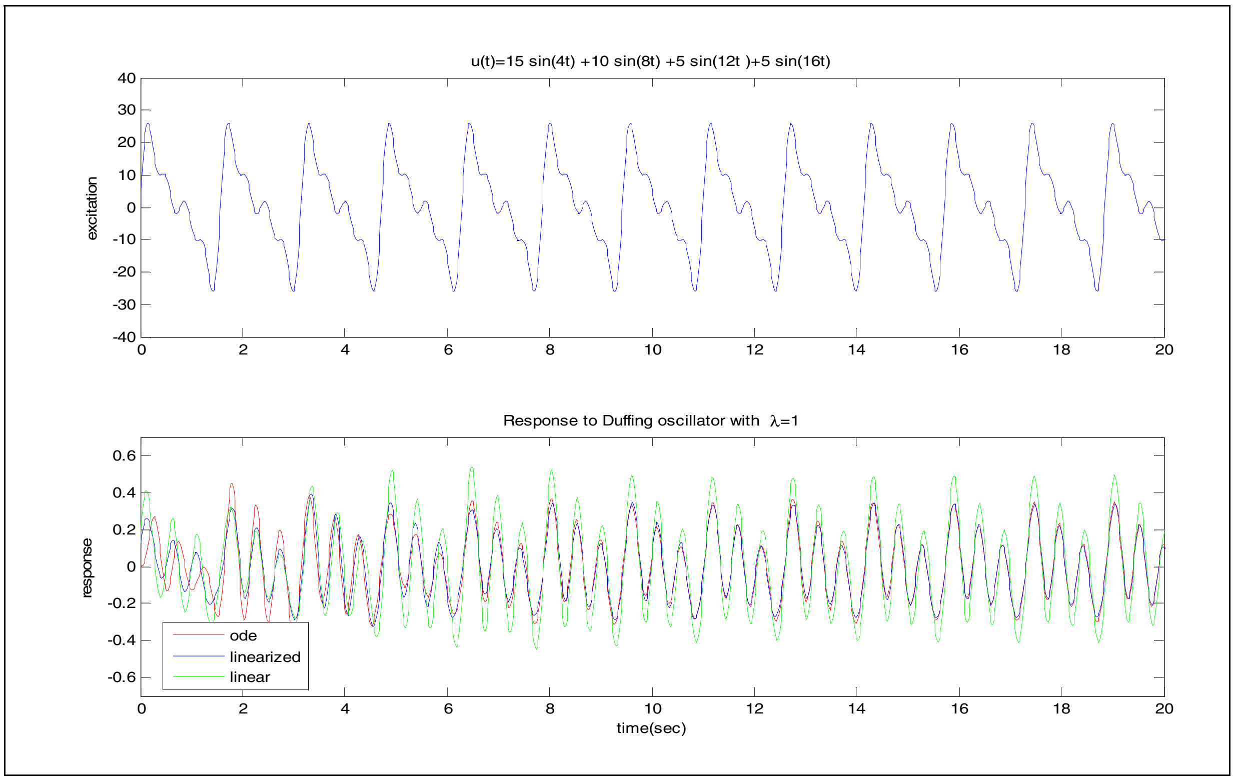

Figure 1. Equivalent Linearization for Stationary Input

The method described above works well for systems subjected to stationary processes. Since real signals are nonstationary, a modified version of the linearization procedure to account for the time dependence of the system response is proposed. For each frequency band, wavelet coefficients of the response are calculated by the method described by Newland, using the generalized wavelet scheme (Newland, 1997). Exploiting the direct relationship between the absolute values of the Fourier coefficients of the response and its wavelet coefficients, scale dependent equivalent stiffness values are calculated as,

$$
\begin{aligned}
k_{nl.m} &= 3\pi^2 (n-m)^2 \lambda\ k\ A_m^{\ 2} \left[1 + 2\left(\frac{A_{m+1}}{A_m}\right)^2 + 2\left(\frac{A_{m+2}}{A_m}\right)^2 + \ldots\ldots + 2\left(\frac{A_{n-1}}{A_m}\right)^2 \right] \\
k_{nl.m+1} &= 3\pi^2 (n-m)^2 \lambda\ k\ A_{m+1}^{\ 2} \left[2\left(\frac{A_m}{A_{m+1}}\right)^2 + 1 + 2\left(\frac{A_{m+2}}{A_{m+1}}\right)^2 + \ldots + 2\left(\frac{A_{n-1}}{A_{m+1}}\right)^2 \right] \\
&\vdots \\
&\vdots \\
k_{nl.n-1} &= 3\pi^2 (n-m)^2 \lambda\ k\ A_{n-1}^{\ 2} \left[2\left(\frac{A_m}{A_{n-1}}\right)^2 + 2\left(\frac{A_{m+1}}{A_{n-1}}\right)^2 + 2\left(\frac{A_{m+2}}{A_{n-1}}\right)^2 \ldots\ldots + 1 \right]
\end{aligned}
\qquad (14)
$$

and the transfer function is updated accordingly. In Equation(14), A_i represents the i^{th} component of the Fourier transform of the wavelet coefficients of the response, As one can see from Eq.(14), for scale (m,n), there are (n-m) equivalent stiffness values, each 1/(n-m) units apart on the time axis. The process is iterated until the solution converges. It was seen that convergence is usually achieved in a few steps.

To demonstrate its validity, the procedure described above has been applied to the system defined in Eq.(13) and the steady state system response from the linearization is compared to that from the numerical analysis (4^{th} order Runge Kutta algorithm) in Figure(2). Here, the excitation is a recorded accelerogram from the Kocaeli, Turkey earthquake (8/17/1999).

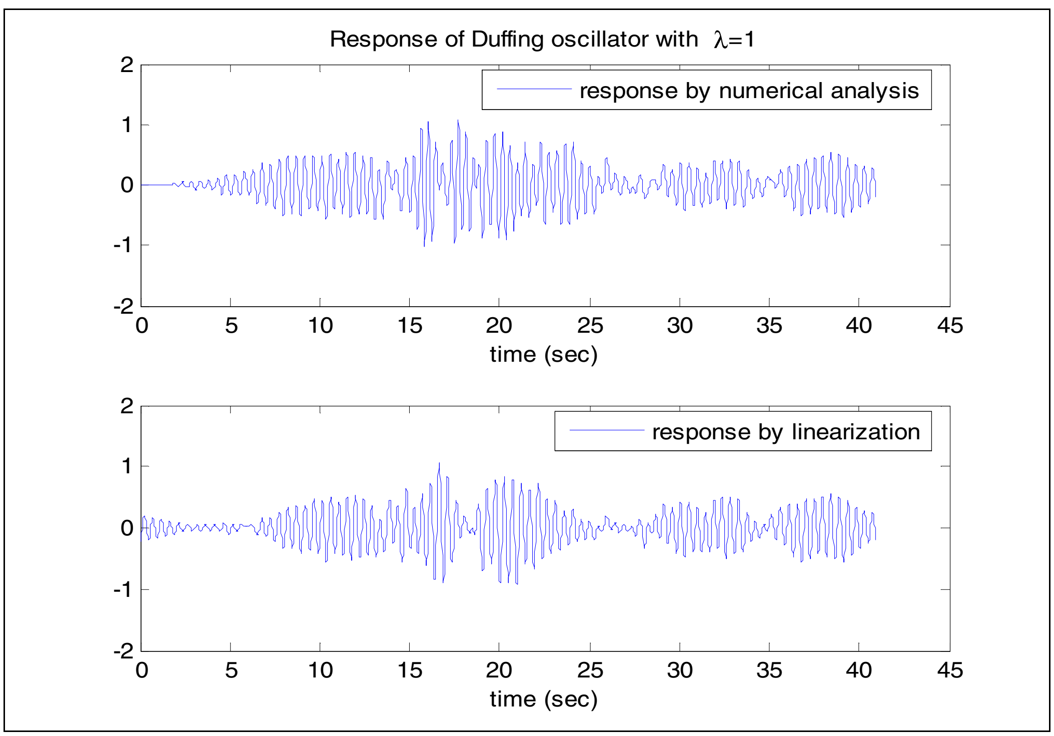

Figure 2. a) Numerical solution b)Wavelet based solution for λ = 1

Conclusions

A numerical approach for determining steady state response for a system with cubic nonlinearity in stiffness is proposed. Generalized harmonic wavelets are used to define scale dependent equivalent stiffness values. Starting from the frequency response function of the linear system, an iterative method is applied to update the equivalent stiffness values such that the mean square error between the nonlinear system and its linear equivalent is minimized. No assumptions have been made regarding the probability distribution of the excitation or the response. Numerical results pertaining to the response of an oscillator with cubic nonlinearity in stiffness with nonlinearity parameter $\lambda=1$ have been presented. It is concluded that the numerical approach developed in this paper gives a reliable estimate of the steady state response of systems with cubic nonlinearity in stiffness, when subjected to nonstationary excitations. Future research will focus on the applicability of this procedure on systems with hysteretic nonlinearity.

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