Analysis and Preliminary Experimental Study on Central Difference Method for Real-time Substructure Testing

B. Wu, Q. Wang, H. Bao and J. Ou

ABSTRACT

Central difference method (CDM) that is explicit for pseudo dynamic testing is also supposed to be explicit for real-time substructure testing (RST). However, to obtain correct velocity dependent restoring force of the physical substructure being tested, the target velocity is required to be calculated as well as displacement. The standard CDM provides only explicit target displacement but not explicit target velocity. This paper investigates the necessary modification of standard central difference method when applied to RST and analyzes the stability of the modified CDM for RST (CDM-RST). The analysis shows that the stability of the CDM-RST decreases with increasing damping ratio of the physical substructure. Then a preliminary experimental research is described. The test shows that the calculated result agrees well with the tested one when the damping ratio of the specimen (i.e., damper) is relatively low, but the discrepancy between the tested and calculated responses increases with the increasing damping ratio of the specimen.

Key words: real-time, substructure testing, central difference method, stability

1. INTRODUCTION

The pseudo-dynamic testing (PDT) is an experimental technique for simulating the earthquake response of structures and structural components in the time domain. In this test, the structural system is represented as a discrete spring-mass system, and its dynamic response to earthquakes is solved numerically using direct integration. Unlike conventional direct integration algorithms, in the pseudo-dynamic test the restoring forces of the system are not modeled but are directly measured from a test conducted in parallel with the direct integration. In many structures, the unpredictable nonlinear behavior that provides the motivation for laboratory testing is quite localized. In these circumstances a far more economical test can be performed using the pseudo-dynamic testing with substructuring approach or pseudo-dynamic substructure testing (PST). The algorithms and implementations of PDT and PST are well documented, e.g., Mahin & Shing (1985), Takanashi & Nakashima (1987).

One of the critical prerequisites for conducting PDT is that the effect of the loading rate on the restoring force of the structure should be of minor consequence, because the structure is loaded quasi-statically in the PDT. Lately, a variety of new types of structural components and devices have been introduced in structures, particularly in connection with their vibration control (Soong and Spencer, 2002). Many of them are very velocity dependent in vibration characteristics such as...
viscous dampers or viscoelastic dampers. To test the velocity dependent components incorporated in structures, real-time substructure testing (RST) was developed in 1990s. The first reported RST test (Nakashima et al. 1992) was performed on a viscous damper located at the base of a multi-story building. Only the damper was tested physically, with the isolated building modeled numerically.

A key element of the RST as well as PDT is the numerical algorithm that is used to perform the stepwise integration of the equations of motion. Many numerical algorithms have been used in RST such as central difference method (Nakashima et al. 1992, Nakashima and Masaoka 1999, Darby et al. 1999, 2001, Horiuchi et al. 1999, Horiuchi and Konno 2001), linear acceleration method (Horiuchi et al. 2000), backward Euler method (Igarashi 2002), Tustin’s method (Blakeborough et al. 2001), and first-order-hold discretization method (Darby et al. 2001). Central difference method (CDM), which is explicit for PDT is also believed explicit for RST (Williams and Blakeborough, 2001). However, to obtain correct velocity dependent restoring force of the physical substructure being tested, the calculated target velocity is required as well as displacement. The standard CDM provides only explicit target displacement but not explicit target velocity. This paper will investigate the required modification of standard central difference method when applied to RST and analyze the stability of the modified central difference method of RST (CDM-RST), and then will describe the preliminary experimental study on a structure with a viscous damper.

2. CENTRAL DIFFERENCE METHOD FOR RST (CDM-RST)

For RST, the equations of motion may be written in matrix form as

$$M_N \ddot{X} + R_N (X, \dot{X}) + R_E (X, \dot{X}) = F$$

(1)

where $M_N$ is the mass matrix of the numerical substructure, $R_N$ restoring force vector of the numerical substructure, $R_E$ restoring force vector of the physical substructure (test specimen), $X$ the vector of nodal displacements, $F$ the vector of external excitation forces, and dots represent differentiation with respect to time. In many substructure tests the mass of the specimens can be ignored and the properties of the specimens are not related to acceleration so that the restoring forces take the form of $R_E (X, \dot{X})$. Then equation (1) becomes

$$M_N \ddot{X} + R_N (X, \dot{X}) + R_E (X, \dot{X}) = F$$

(2)

We assume that the numerical substructure is with linear damping force and displacement dependent restoring force, i.e.

$$R_N (X, \dot{X}) = C_N \dot{X} + R_N (X)$$

(3)

where $C_N$ is the damping coefficient of numerical substructure. Substituting equation (3) into (2), we get

$$M_N \ddot{X} + C_N \dot{X} + R_N (X) + R_E (X, \dot{X}) = F$$

(4)

Using the CDM, the velocity and acceleration in step $i$ are approximated by

$$\dot{X} = \frac{X_{i+1} - X_{i-1}}{2\Delta t}$$

(5)

$$\ddot{X} = \frac{X_{i+1} - 2X_i + X_{i-1}}{\Delta t^2}$$

(6)

where $\Delta t$ is time interval. Substituting equations (5) and (6) into equation (4) at $i$th step, we obtain
\[ X_{i+1} = \left( \frac{M_N}{\Delta t^2} + \frac{C_N}{2\Delta t} \right)^{-1} \left[ F_i - R_N(X_i) + 2\frac{M_N}{\Delta t^2} X_i - \left( \frac{M_N}{\Delta t^2} - \frac{C_N}{2\Delta t} \right) X_{i-1} - R_E(X_i, \dot{X}_i) \right] \]  

(7)

From the above equation we see that the calculation of \( X_1 \) involves \( X_0 \) and \( X_{-1} \). With \( X_0, \dot{X}_0, \ddot{X}_0 \) known (given \( X_0 \) and \( \dot{X}_0 \), \( \ddot{X}_0 \) is calculated using equation (4)), the relations in equations (5) and (6) can be used to obtain \( X_{-1} \) as (Bathe. 1996)

\[ X_{-1} = X_0 - \Delta t \dot{X}_0 + \frac{\Delta t^2}{2} \ddot{X}_0 \]  

(8)

In conventional PDT, the calculated target displacement is imposed upon the specimen and then the rate independent restoring force can be measured. For RST, the velocity of the next step (i.e. step \( i+1 \)) must also be calculated and imposed on the specimen to obtain correctly the restoring force dependent on velocity. However, with current CDM represented by equations (4)-(6), the velocity of step \( i+1 \) cannot be calculated explicitly. To the writers’ knowledge, the issue of the velocity calculation method in RST and particularly its consequence on the stability and accuracy have not been discussed theoretically by other researchers.

To achieve sufficient accuracy in both displacement and velocity control, a digital servo-mechanism was used in the RST by Nakashima et al. (1992). The mechanism interpolates the target displacement signal \( X_{i+1} \) linearly into a set of displacement signals: \( X_{i+2}, X_{i+1}, \ldots, j X_{i+1}, \ldots, j_{0} X_{i+1} \) with \( j X_{i+1} = X_i + j \times \Delta \times \dot{X}_i \) and \( j_{0} X_{i+1} = X_{i+1} \). In other words, the following additional assumption for the target velocity was implied in Nakashima et al. (1992)’s test:

\[ \dot{X}_{i+1} = \frac{X_{i+1} - X_i}{\Delta t} \]  

(9)

With the above equation, CDM becomes explicit for velocity as well as for displacement.

3. STABILITY AND ACCURACY ANALYSIS OF CDM-RST

To analyze the stability and accuracy, we consider single-degree-of-freedom (SDOF) system with linear numerical and physical substructures, i.e.,

\[ R_N(X) = K_N X \]  

(10)

\[ R_E(X, \dot{X}) = C_E \dot{X} + K_E X \]  

(11)

in which \( K_N \) is stiffness of the numerical substructure, \( C_E \) and \( K_E \) are damping coefficient and stiffness of physical substructure, respectively. Substituting equations (10) and (11) into (7) and then (7) into (9), we obtain

\[ X_{i+1} = \left( \frac{M_N}{\Delta t^2} + \frac{C_N}{2\Delta t} \right)^{-1} \left[ F_i - \left( K_N + K_E - 2\frac{M_N}{\Delta t^2} \right) X_i - \left( \frac{M_N}{\Delta t^2} - \frac{C_N}{2\Delta t} \right) X_{i-1} - C_E \dot{X}_i \right] \]  

(12)

\[ \dot{X}_{i+1} = \left( \frac{M_N}{\Delta t^2} + \frac{C_N}{2} \right)^{-1} \left[ F_i - \left( K_N + K_E - \frac{M_N}{\Delta t^2} + \frac{C_N}{2\Delta t} \right) X_i - \left( \frac{M_N}{\Delta t^2} - \frac{C_N}{2\Delta t} \right) X_{i-1} - C_E \dot{X}_i \right] \]  

(13)

The stability and accuracy can be evaluated with the free vibration solution succinctly written in the recursive form (Shing and Mahin, 1985)

\[ Y_{i+1} = \Lambda Y_i \]  

(14)

where
and the amplification matrix of CDM-RST, $A$, is expressed as

$$
A = \begin{bmatrix}
    \frac{2 - \Omega^2}{1 + \xi_N \Omega} & \frac{\xi_N \Omega - 1}{1 + \xi_N \Omega} & -\frac{2 \xi_E \Omega}{1 + \xi_N \Omega} \\
    0 & \frac{\xi_N \Omega - 1}{1 + \xi_N \Omega} & -\frac{2 \xi_E \Omega}{1 + \xi_N \Omega} \\
    -\frac{\xi_N \Omega - \Omega^2}{1 + \xi_N \Omega} & \frac{\xi_E \Omega}{1 + \xi_N \Omega} & \frac{\xi_N \Omega - 1}{1 + \xi_N \Omega}
\end{bmatrix}
$$

in which $\Omega = \Delta t \omega = \Delta t \sqrt{\left(K_N + K_E\right) / M_N}$,  $\xi_N = C_N / (2M_N \omega)$,  $\xi_E = C_E / (2M_N \omega)$. Stability and accuracy of an algorithm depend upon the eigenvalues of amplification matrix.

### 3.1 Stability

The stability condition of an integration method is (Shing and Mahin, 1985)

$$
\rho(A) \leq 1 \tag{17}
$$

where $\rho(A)$ is spectral radius of $A$ which is defined as max|$\lambda_i$|, and $\lambda_i$ is eigenvalues of $A$. For the matrix $A$ in equation (16), we have

$$
\lambda_{i,2} = A \pm iB \text{ and } \lambda_3 = 0 \tag{18}
$$

where

$$
A = \frac{2 - \Omega^2 - 2 \xi_E \Omega}{2(1 + \xi_N \Omega)}, \quad B = \frac{\Omega \sqrt{-\Omega^2 - 4 \xi_E \Omega + 4 - 4 \xi_N^2 - 4 \xi_E^2 - 8 \xi_E \xi_N}}{2(1 + \xi_N \Omega)} \tag{19}
$$

when

$$
\Omega < \sqrt{4 - 4 \xi_N^2 - 8 \xi_N \xi_E - 2 \xi_E} \tag{20}
$$

or

$$
\lambda_{i,2} = A \pm B \text{ and } \lambda_3 = 0 \tag{21}
$$

where

$$
A = \frac{2 - \Omega^2 - 2 \xi_E \Omega}{2(1 + \xi_N \Omega)}, \quad B = \frac{\Omega \sqrt{\Omega^2 + 4 \xi_E \Omega - 4 + 4 \xi_N^2 + 4 \xi_E^2 + 8 \xi_E \xi_N}}{2(1 + \xi_N \Omega)} \tag{22}
$$

when

$$
\Omega \geq \sqrt{4 - 4 \xi_N^2 - 8 \xi_N \xi_E - 2 \xi_E} \tag{23}
$$

From inequalities or equations (17)-(23), the stability criteria for CDM-RST can be obtained as

$$
\Omega > \sqrt{4 + 4 \xi_E^2 - 2 \xi_E}, \text{ Unstable} \tag{24a}
$$

$$
\sqrt{4 - 4 \xi_N^2 - 8 \xi_N \xi_E - 2 \xi_E} \leq \Omega \leq \sqrt{4 + 4 \xi_E^2 - 2 \xi_E}, \text{ Stable (two real and one zero eigenvalues)} \tag{24b}
$$

$$
\Omega < \sqrt{4 - 4 \xi_N^2 - 8 \xi_N \xi_E - 2 \xi_E}, \text{ Stable (two complex conjugate and one zero eigenvalues)} \tag{24c}
$$

From inequality (24a), we see that unlike CDM for PDT (CDM-PDT)(Nakashima, 1985), the upper limit of $\Omega$ for a stable CDM-RST is not constant, and that the stability limit of $\Omega$ decreases with increasing damping ratio of the physical substructure, which means the stability of CDM-RST...
deteriorates with higher damping ratio of the specimen. When the damping ratio of the physical substructure is zero, the stability limit becomes 2 that is the same as the result of the CDM-PDT.

### 3.2 Accuracy

The details the accuracy analysis of the algorithm is referred to Wu el al.(2004). Only the main result are summarized here as follows: (1) numerical damping ratio is positive and increases both with increasing $\omega\Delta t$ and with increasing damping ratio of physical substructure for most $\omega\Delta t$ in the stable range; minor negative numerical damping ratio occurs when $\omega\Delta t$ is near stable limit for the cases with relatively low damping ratio of physical substructure; (2) period distortion increases with increasing $\omega\Delta t$ and damping ratio of physical substructure except for very high damping ratio of physical substructure; and (3) the initial velocity is twisted and the amount of twisting increases with increasing $\omega\Delta t$ and damping ratio of physical substructure.

### 4. PRELIMINARY EXPERIMENTAL STUDY

#### 4.1 Test Setup

The tests were carried out at Mechanical and Structural Testing Center, Harbin Institute of Technology (HIT). The whole structure is a single story frame structure incorporated with a viscous damper, of which the bare frame without the damper was the numerical substructure, and the damper was the physical substructure. The schematic of the structure is shown in Figure 1. The Schenck servohydraulic actuator was managed and controlled by MTS software system. The test setup is shown in Figure 2.

#### 4.2 Test Program

A series of preliminary tests using RST technique have been done at HIT. The parameters of two cases are listed in Table 1. The excitation is El Centro (NS, 1940) earthquake wave. In this preliminary experimental research, we didn't divide the target displacement into several parts and send them successively in order to achieve the target velocity as
Nakashima et al. (1992). The hysteresis behavior of the damper was tested previously by Long (2004). The viscous damping factor of the damper was 153kNs/m, and a linear model very well agreed with the test results (Long, 2004). The damping ratios of the damper in Table 1 are calculated by using this tested viscous damping factor.

Table 1. Test Case

<table>
<thead>
<tr>
<th>Case</th>
<th>$M_N$ ($10^3$kg)</th>
<th>$K_N$ (kN/m)</th>
<th>$\xi_N$</th>
<th>$\xi_E$</th>
<th>$\Delta t$ (s)</th>
<th>$\Omega$</th>
<th>Peak acc. of excitation (m/s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>122</td>
<td>4800</td>
<td>5%</td>
<td>10%</td>
<td>0.01s</td>
<td>0.0628</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>61</td>
<td>2400</td>
<td>0</td>
<td>20%</td>
<td>0.01</td>
<td>0.0628</td>
<td>0.6</td>
</tr>
</tbody>
</table>

4.3 Test Results

The test results of case 1 with damping ratio 10% of the physical substructure is shown in Figure 3. The displacement command calculated based on tested damper force is designated "tested", and the displacement calculated using the previously tested damping ratio, i.e., 153kNs/m, is designated "calculated". From Figure 3 we see that the calculated displacement matches very well with the tested one; the calculated damper force also agrees well with the tested one except at around 8s and 9.5s when there are some measuring noises. The test results of case 2 with damping ratio 20% of the physical substructure is shown in Figure 4. The discrepancies between calculation and test increase due to the increase of the damping ratio of the physical substructure. The target velocity of the damper was not guaranteed because only target displacement was imposed on the specimen. Then an error between the actual velocity and calculated velocity was inevitable. The larger the damping ratio of the specimen, the larger the disagreement between the calculated and tested responses would be.
Figure 3. Test results of Case 1 ($\xi_N=5$, $\xi_E=10\%$, $\Omega=0.0628$)

(b) Damper force

(a) Displacement
5. CONCLUSIONS

To maintain the explicit form of CDM both for velocity and for displacement, some modification on the algorithm is required when it is applied to RST. The stability of a modified CDM for RST with additional assumption about target velocity is investigated. The analysis result shows that the stability of CDM-RST deteriorates as the damping ratio of physical substructure increases. A preliminary experimental research is carried out on a single story structure incorporating a viscous damper. The test result shows that the calculated result agrees well with the tested one when the damping ratio of the specimen (i.e., damper) is relatively low, but the discrepancy between the tested and calculated responses increases with increasing damping ratio of the specimen.

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REFERENCE


