Method of Nonlinear Dynamic Response Analysis Based on Pattern of Self-Equilibrating Stresses*

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ABSTRACT

In this paper, a new method of nonlinear dynamic response analysis of structures based on a pattern of self-equilibrating stresses is given. The theory and procedure of analysis are described in detail. This method can be used to solve the nonlinear dynamic response of multi-story and high rise frame structures, shear-wall structures, frame-shear wall structures and multi-story brick structures modeled by different elements with different hysteretic models. The corresponding examples show that this method is effective and reliable. Especially, the method has great advantage in dealing with the softening feature of negative stiffness.

Key words: self-equilibrating stresses, nonlinear, dynamic response analysis

INTRODUCTION

In recent years, people have recognized more and more clearly that the effective analysis method of nonlinear response is very significant for seismic design and research of structures, because it can provide actual condition of the nonlinear response of structures. But the nonlinear dynamic response analysis is a very difficult problem. At present, the common analytical methods are direct integration methods, which include central difference method, Houbolt method, Wilson-θ method and Newmark method. In the central difference method, the solution at time \( t + \Delta t \) is calculated by using the equilibrium conditions at time \( t \). For this reason, the integration procedure is called an explicit integration method, and it is noted that such integration schemes do not require solving equation and to inverting the stiffness matrix in case of neglecting the damping matrix or only considering mass dependent damping. But the central difference method is conditionally stable, so the time step \( \Delta t \) used is required smaller than a critical time step \( \Delta t_c \). That means the number of integration and the computation time are increased significantly. In Houbolt method, Wilson-θ method and Newmark method, the equilibrium equation is considered at time \( t + \Delta t \). For this reason, these methods are an implicit integration scheme. In analysis, they require solving nonlinear equation, and thus the stiffness matrix is required to revised and inverted. But the time step \( \Delta t \) used in these methods can be more larger than one used in the explicit integration methods in case of meeting the needs of precision. So the number of integration and the computation time are decreased. When direct integration methods are used to solve the nonlinear equations, usually we use the iteration methods (including initial strain and initial stress methods) or variable stiffness methods to deal with the problem of variable stiffness. The iteration methods need a lot of iteration in the point of variable stiffness. In the nonlinear dynamic response analysis for building structures, specially for reinforced concrete structures, the hysteretic models are in the shape of multi-linear-line, the ratio of plastic and elastic stiffnesses is very small and the unloading and reloading stiffnesses change suddenly after yielding, so during the iteration process it is usual to meet the problems of slow convergence or divergence. There are some methods for dealing with these problems, but these methods still require a lot of time. So in analysis of nonlinear response of building, the variable stiffness methods are used usually, and

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the methods often used are Wilson-θ method and Newmark method. The central difference method is used less because it is conditionally stable and requires the time step $\Delta t$ being very small. The variable stiffness methods need to revise and invert the stiffness matrix when the stiffness changes. So these methods also require spending a lot of computation time, especially for the dynamic analysis of super high-rise buildings and three dimensional structures with huge degree of freedom, furthermore the computers with large capacity are needed, therefore it is not convenient to use in practice. Since the 1950s, there were some researchers studying on the structural elastoplastic theory and analysis method based on a pattern of self-equilibrating stresses, and using it to evaluate the collapse load and elastoplastic deformation at incipient collapse. In 1980s, some researchers [3, 4] used this theory for nonlinear static response analysis of structure and achieved very good results. The difference between this new method and the classical methods is that rather than updating and inverting the stiffness matrix, a pattern of self-equilibrating stresses caused by plastic hinges is computed instead. Owing to not revising the stiffness matrix, this method is faster than the classical methods in computation, and can save time significantly. The primary objective of this paper is to develop a new method of nonlinear dynamic response of buildings using the elastoplastic theory based on a pattern of self-equilibrating stresses.

Firstly, the theory of elastoplastic response analysis based on a pattern of self-equilibrating stresses is introduced. Secondly, this theory is extended to the nonlinear dynamic response analysis, and the analytical procedure and theoretical formulas are given. And lastly, two examples are shown to illustrate the application of this method.

THE THEORY OF ELASTOPLASTIC RESPONSE ANALYSIS BASED ON A PATTERN OF SELF-EQUILIBRATING STRESSES

The thinking of the elastoplastic theory based on a pattern of self-equilibrating stresses is that when one or several critical sections attain their plastic moment capacity, the elastic limit is reached and the plastic hinges are formed. The formation of these plastic hinges changes the stiffness matrix of the entire structure, and causes the redistribution of stresses of structure. In order to continue the elastoplastic analysis, rather than updating the stiffness matrix as the classical methods, the rotation at plastic hinge is regarded as an external load to apply to the original structure instead. Then a pattern of self-equilibrating stress distribution, which represents the redistribution of stresses, resisting the plastic rotation at plastic hinges is computed. So, the structure is restored to the original state before the rotation of plastic hinge, and the updating and inverting the stiffness matrix is avoided. For this reason, the procedure of calculation is simplified and the time is saved.

Firstly, the inelastic constitutive laws, compatibility and equilibrium conditions for the assembled structure, which are derived assuming the associated flow rule, the piecewise-linearized yield condition, and the small-displacement theory, are needed to state in the analysis based on this elastoplastic theory. These relationships are introduced briefly, and the detail can be seen in reference [5].

The establishment of the basic relationship of this elastoplastic theory is in relation to both the plastic function and plastic multiplier. The plastic function is defined as the difference between the structural response and the plastic capacity. For the elastic-perfectly plastic moment-curvature relationship of beam section $i$, the plastic function is expressed as:

$$\Phi^+ = M_i - R_i^+ \leq 0 \quad \text{and} \quad \Phi^- = -M_i - R_i^- \leq 0$$

(1)
where \( R_i^+ \) and \( R_i^- \) represent the plastic moment capacity of section \( i \); \( \Phi_i^+ = 0 \) (or \( \Phi_i^- = 0 \)) indicates that the positive (or the negative) yield mode has been activated. The plastic multipliers are quantities that measure the intensity of the activation of the yield surfaces, i.e. the level of plastic deformations, their expressions are:

\[
\lambda_i^+ \geq 0 , \quad \lambda_i^- \geq 0 
\]  

(2)

\[\begin{align*}
\text{Figure 1 Elastic-perfectly plastic} \\
\text{moment-curvature relationship of} \\
\text{beam cross section } i
\end{align*}\]

\[\begin{align*}
\text{Figure 2 Elastic-linear workhardening} \\
\text{moment-curvature relationship of} \\
\text{beam cross section } i
\end{align*}\]

The plastic elongation \( \theta_i^p \) shown in Figure 1 can be expressed in terms of the plastic multipliers \( \lambda_i \):

\[
\theta_i^p = \lambda_i^+ - \lambda_i^- 
\]  

(3)

It is obvious that a plastic multiplier has a positive value (\( \lambda_i > 0 \)) only if the particular yield mode is activated (\( \Phi_i^+ = 0 \)), and it must be zero (\( \lambda_i^- = 0 \)) when the element is in the elastic range (\( \Phi_i^+ < 0 \)). This relationship between the plastic multiplier (\( \lambda \)) and the plastic function (\( \Phi \)) represents the hypothesis of the holonomic (reversible) behavior, which implies that the analysis is independent of the loading history, and can be condensed as follows:

\[
\Phi_i^+ \lambda_i^+ = 0 \quad , \quad \Phi_i^- \lambda_i^- = 0 
\]  

(4)

Now, let us consider the behavior of section \( i \) when the moment-curvature relationship has been approximated by elasto-linear work-hardening behavior as shown in Figure 2. If we introduce a hardening parameter, \( H \), for each yield mode such that \( H_i^+ \lambda_i^+ \) measures the extra resistance sustained as a result of activation of yield mode 1, the yield functions will read:
\[ \Phi_1^+ = M - R_i^+ - H_i^+ \lambda_i^+ \leq 0 \]
\[ \Phi_2^+ = M - R_i^+ - H_i^+ \lambda_i^+ \leq 0 \]
\[ \Phi_1^- = -M - R_i^- - H_i^- \lambda_i^- \leq 0 \]  
(5)

The hardening parameters \( H_i^+ \), \( H_i^- \) and \( H_i^0 \) can be determined from the stiffnesses corresponding to each state of hysteretic model, \( k_i^+ \), \( k_i^- \) and the elastic stiffness \( k_0 \) (see Appendix A).

If we express equation (1) to (5) in vector forms, and describe the entire response deformation as the sum of the elastic deformation \( \theta_i^e \) and the plastic deformation \( \theta_i^p \), the constitutive laws of holonomic behavior for a single stress component can be expressed as follows:

\[
\{ \Phi_i \} = [n_i] \cdot M_i - [H_i] \{ \lambda_i \} - \{ R_i \} \\
\{ \Phi_i \} \leq \{ 0 \}, \quad \{ \lambda_i \} \geq \{ 0 \}, \quad \{ \Phi_i \} \cdot \{ \lambda_i \} = 0 \\
\theta_i^p = [n_i] \{ \lambda_i \}, \quad \theta_i^e = k_i^{-1} M_i, \quad \theta_i = \theta_i^e + \theta_i^p
\]  
(6)

where the vector \([n_i]\) represents the outward unit normal vector to the yield planes; \( k_i \) represents the elastic stiffness.

If we imitate the concept of a single stress component, the constitutive laws for two stress components and multi-stress components are derived as follows:

\[
\{ \Phi_i \} = [n_i] \cdot \{ S_i \} - [H_i] \{ \lambda_i \} - \{ R_i \} \\
\{ \Phi_i \} \leq \{ 0 \}, \quad \{ \lambda_i \} \geq \{ 0 \}, \quad \{ \Phi_i \} \cdot \{ \lambda_i \} = 0 \\
\{ p_i \} = [n_i] \{ \lambda_i \}, \quad \{ e_i \} = [k_i]^{-1} \{ S_i \}, \quad \{ s_i \} = \{ e_i \} + \{ p_i \}
\]  
(7)

where \( \{ S_i \} \) represents the stress resultant vector; \( \{ s_i \} \) represents the total strain vector; \( \{ e_i \} \) represents the elastic strain vector; \( \{ p_i \} \) represents the plastic strain vector, respectively.

In the previous sections, the constitutive laws were expressed as a function of the active stress resultant and active deformation at a cross section. This is essential that the constitutive laws are also needed to express in terms of the natural stress resultant and deformation because the compatibility and equilibrium conditions are generally expressed in terms of natural quantities.

We can transform the active quantities to the natural quantities by introducing the static transformation matrix, \([B]\):

\[
\{ S \} = [B] \{ Q \} \\
\{ q \} = [B] \cdot \{ s \}
\]  
(8)

For a structure composed of \( m \) elements, the constitutive laws for the (unassembled) structure can be obtained by extending the above constitutive laws over all elements:
\[
\{\Phi\} = [N]^t\{Q\} - [H]\{\lambda\} - \{R\} \quad (9a)
\]
\[
\{\Phi\} \leq 0, \quad \{\lambda\} \geq \{0\}, \quad \{\Phi\}'\{\lambda\} = 0 \quad (9b,c,d)
\]
\[
\{p\} = [N]\{\lambda\}, \quad \{e\} = [k]^{-1}\{Q\}, \quad \{q\} = \{e\} + \{p\} \quad (9e,f,g)
\]

where the vectors \{Q\}, \{q\}, \{e\}, \{p\}, \{\Phi\}, \{\lambda\} and \{R\} contain the corresponding vectors for the \(m\) individual frame elements in a particular order, and \{N\}, \{H\} and \{k\} are the block-diagonal matrices of the element matrices \([N^e] = [B^e]^t\{n^e\}, \quad [H^e] \quad \text{and} \quad [k^e]\). The equations (9a,b) express the yield condition of the entire structure; and the equations (9c)~(9e) are the plastic flow rule of the piecewise-linearized yield condition.

For the assembled structure, the global displacement vector, \{u\}, and the corresponding external load vector, \{F\}, identified at the nodal degrees of freedom are related to the natural (element) deformation vector, \{q\}, and the natural (element) stress vector, \{Q\}, through the compatibility matrix, \{C^t\}. The compatibility and equilibrium conditions for the assembled structure can be obtained as follows:

\[
\{q\} = \{C^t\}\{u\} \quad \quad \quad \quad (10)
\]
\[
\{C^t\}'\{Q\} = \{F\} \quad \quad \quad \quad (11)
\]

Equations (9)~(11) fully govern the elastoplastic holonomic behavior of the structure. These are the theory and formulas of the nonlinear holonomic analysis of the elastoplastic structure based on a pattern of self-equilibrating stresses.

Figure 3 The nonholonomic elastic-linear workhardening behavior of beam cross section i

For nonholonomic behavior that implies that the analysis is dependent on the loading history, its analytical theory and formulas can be dealt with by using the same methods as the holonomic analysis, but be expressed by increment form. Now, the relationship between the plastic potential function and the plastic multiplier is shown in Figure 3. When the stress point \(p\) is on a particular yield mode \(j(\Phi_j = 0)\), a small increase in applied loads will either increase the activation intensity by \(\Delta\lambda_j \geq 0\) such that the stress point remains on the yield mode \(j\), i.e. \(\Delta\Phi_j = 0\), or the yield surface \(j\) is
unloaded in which case $\Delta \Phi_j < 0$ and $\Delta \lambda_j = 0$. In other words, the nonholonomy hypothesis requires that:

$$\Delta \Phi_j \Delta \lambda_j = 0 \quad \text{and} \quad \Phi_j \Delta \lambda_j = 0$$

(12)

The second relationship is required to show that the stress point must be on the particular yield mode ($\Phi_j = 0$) in order to increase the activation intensity ($\Delta \lambda_j > 0$). The first relation simply states that once a yield surface has been activated, it is possible either to increase the activation intensity ($\Delta \lambda_j > 0$, in which case $\Delta \Phi_j = 0$) or to unload the yield mode ($\Delta \Phi_j < 0$, in which case $\Delta \lambda_j = 0$). Therefore, the constitutive laws for the unassembled structure can be obtained as follows:

$$\{\Delta \Phi\} = [N]^t \{\Delta Q\} - [H]\{\Delta \lambda\}, \quad \{\Phi\} + \{\Delta \Phi\} \leq \{0\} \quad \text{(13a,b)}$$

$$\{\Delta \lambda\} \geq \{0\}, \quad \{\Delta \Phi\}^t \{\Delta \lambda\} = 0, \quad \{\Phi\}^t \{\Delta \lambda\} = 0 \quad \text{(13c,d,e)}$$

$$\{\Delta \rho\} = [N]\{\Delta \lambda\}, \quad \{\Delta \epsilon\} = [k]^{-1}\{\Delta Q\}, \quad \{\Delta \gamma\} = \{\Delta \epsilon\} + \{\Delta \rho\} \quad \text{(13f,g,h)}$$

The compatibility and equilibrium relations for the assembled structure become:

$$\{\Delta \rho\} = [C]^t \{\Delta u\} \quad \text{(14)}$$

$$[C]^t \{\Delta Q\} = \{\Delta F\} \quad \text{(15)}$$

It is important to emphasize that the nonholonomic constitutive laws are only essential for the active yield modes. For the remaining (inactive) yield modes, the holonomy hypothesis applies. This is the reason that when a yield surface is inactive, the stress point lies in the elastic range of that yield mode and the elastic behavior is generally assumed to be reversible.

**NONLINEAR DYNAMIC RESPONSE ANALYSIS**

For the analysis of nonlinear dynamic response of structures, the nonholonomy analysis expressed by the increment form is used. Firstly, the equation of dynamic equilibrium is expressed in terms of the increment form of nodal displacement by using the step by step integration method. Then the equation is related to the elastoplastic constitutive laws, the compatibility and equilibrium conditions for nonholonomic analysis to give the relationship of dynamic nonholonomic behavior and the nodal displacement vector expressed in terms of the external forces and the plastic deformation.

At any instant of time, for a finite time step, $\Delta t$, the equation of dynamic equilibrium can be written as:

$$[M]\{\Delta \ddot{u}\} + [C]\{\Delta \dot{u}\} + [K]\{\Delta u\} = -[M]\{I\}\Delta \ddot{u}_g$$

(16)

where $[M]$ and $[C]$ are the mass matrix and damping matrix; $[K]$ is the stiffness matrix for assembled structure; $\{\Delta \ddot{u}\}$, $\{\Delta \dot{u}\}$ and $\{\Delta u\}$ are the increment vectors of acceleration, velocity and
displacement, respectively, at the nodes; \( \Delta \ddot{u}_g \) is the increment of ground acceleration; \( \{ I \} \) is a unit vector.

By using the constant acceleration method, the increment vectors of velocity and acceleration at the nodes can be written in terms of the displacement vector:

\[
\{ \Delta \dot{u} \} = \frac{2}{\Delta t} \{ \Delta u \} - 2 \{ \ddot{u}_0 (t) \}
\]
\[
\{ \Delta \ddot{u} \} = \frac{4}{\Delta t^2} \{ \Delta u \} - \frac{4}{\Delta t} \{ \dot{u}_0 (t) \} - 2 \{ \ddot{u}_0 (t) \}
\]

(17)

where \( \{ \ddot{u}_0 (t) \} \) and \( \{ \dot{u}_0 (t) \} \) are the vectors of velocity and acceleration at the beginning time of the time step respectively. Substituting the above equations into equation (16), we obtain:

\[
\left( \frac{4}{\Delta t^2} \{ M \} + \frac{2}{\Delta t} \{ C \} + \{ K \} \right) \{ \Delta u \} = \{ \Delta F \}
\]

(18)

Where \( \{ \Delta F \} = - \{ M \} \{ I \} \Delta \ddot{u}_g + \frac{4}{\Delta t} \{ M \} \{ \dot{u}_0 (t) \} + 2 \{ M \} \{ \ddot{u}_0 (t) \} + 2 \{ C \} \{ \dot{u}_0 (t) \} \), is defined as the increment of incipient external load for every time step (in the following it is called as the increment of incipient external load for short).

In the dynamic problems, the external load in the equation of equilibrium (15) is:

\[
\{ \Delta F \} = \{ K \} \{ \Delta u \} = \{ \Delta F \} - \frac{4}{\Delta t^2} \{ M \} \{ \Delta u \} - \frac{2}{\Delta t} \{ C \} \{ \Delta u \}
\]

(19)

Substituting it into the equation (15), we obtain:

\[
\{ C \} ^{\prime} \{ \Delta Q \} = \{ \Delta F \} - \frac{4}{\Delta t^2} \{ M \} \{ \Delta u \} - \frac{2}{\Delta t} \{ C \} \{ \Delta u \}
\]

(20)

However, the element (natural) stress increment vector is:

\[
\{ \Delta Q \} = [k] \{ \Delta e \} = [k] (\{ \Delta q \} - \{ \Delta p \})
\]

(21)

Substituting it into the equation (20), then combining the resulting equation with the compatibility equation, the increment vector of displacement at the nodes is obtained as follows:

\[
\{ \Delta u \} = [K]^{-1} \left( \{ \Delta F \} + \{ C \} ^{\prime} [k] \{ \Delta p \} \right)
\]

(22)
Where $[K] = [K] + \frac{4}{\Delta t^2}[M] + \frac{2}{\Delta t}[C]$ is defined as the incipient stiffness matrix of structure, in which, $[K] = [C']^T[k][C']$ is the assembled structure stiffness matrix. We can see from the above equation that the increment vectors of displacement at the nodes are composed of two components: the one that resists the increment of incipient external load $\{\Delta F\}$,

$$\{\Delta u^{e1}\} = [K]^{-1}\{\Delta F\}$$

and the other one that resists the increment of the plastic deformation $\{\Delta p\}$:

$$\{\Delta u^{p}\} = [K]^{-1}[C']^T[k]\{\Delta p\}$$

Correspondingly, the increment vector of the element deformation $\{\Delta q\}$ and the increment vector of the element stress $\{\Delta Q\}$ can be decomposed into components as well:

$$\{\Delta q\} = [C'][\Delta u] = [C'][K]^{-1}\{\Delta F\} + [C'][K]^{-1}[C']^T[k]\{\Delta p\} - \{\Delta p\} + \{\Delta p\}$$

$$\{\Delta Q\} = [k]\{\Delta e\} = [k][C'][K]^{-1}\{\Delta F\} + [Z]\{\Delta p\} = \{\Delta Q^{e1}\} + \{\Delta Q^{p}\}$$

where $[Z] = [k][C'][K]^{-1}[C']^T[k] - [k]$, is the matrix of plastic influence coefficients. We can prove [5] that the second component of the element stress increment is dynamic self-equilibrating.

Substituting the equation (13f) in the elastoplastic constitutive laws into the above equation, then substituting the resulting equation into the equation (13a) in the elastoplastic constitutive laws, we obtain:

$$\{\Delta \Phi\} = [N]^T\{\Delta Q^{e1}\} + [N]^T[Z][N]\{\Delta \lambda\} - [H]\{\Delta \lambda\}$$

Thus the yield condition (27) together with the equations (13b) through (13e) fully govern the nonlinear dynamic nonholonomic response of the structure:

$$\{\Delta \Phi\} = [N]^T\{\Delta Q^{e1}\} - [A]\{\Delta \lambda\}$$

$$\{\Phi\} + \{\Delta \Phi\} \leq \{0\}, \quad \{\lambda\} \geq \{0\}$$

$$\{\Delta \Phi\}^T\{\Delta \lambda\} = 0, \quad \{\Phi\}^T\{\Delta \lambda\} = 0$$

where $[A] = -[N]^T[Z][N] + [H]$.

We can see from the above analysis that every structural response (the element or nodal response) all can be expressed as two components: the one that resists the increment of incipient external load, and the other one that resists the increment of the plastic deformation. The main difference between the new method of nonlinear dynamic analysis based on a pattern of self-equilibrating stresses and the
classical methods is that the plastic deformation is separated from the elastic one. Besides computing the pattern of dynamic self-equilibrating stresses instead of updating and reverting the stiffness matrix, the analytical procedure of this new method is the same as the classical one basically.

In the linear phase, i.e. before the formation of the plastic hinges, the structure only resists the increment of incipient external load, so the increment vector of displacement at the nodes is equal to the component that resists the increment of incipient external load:

\[
\{ \Delta u \} = \{ \Delta u^{el} \} 
\]  

(29)

where \( \{ \Delta u^{el} \} \) can be computed by the equation (23).

When the structure gets into the nonlinear phase, the plastic hinges are formed. Therefore, the increment vector of displacement at the nodes is composed of two components of resisting the increment of incipient external load and resisting the increment of plastic deformation:

\[
\{ \Delta u \} = \{ \Delta u^{el} \} + \{ \Delta u^p \} 
\]  

(30)

According to the above equations, we know:

\[
\{ \Delta u^p \} = [K]^{-1} [C^r] \cdot [k] \{ \Delta p \} \\
\{ \Delta p \} = [N] \{ \Delta \lambda \}
\]

Substituting the second equation into the first equation, we obtain:

\[
\{ \Delta u^p \} = [K]^{-1} [C^r] \cdot [k] [N] [\Delta \lambda] = [C_p] [\Delta \lambda]
\]  

(31)

where \([C_p] = [K]^{-1} [C^r] \cdot [k] [N] \), is defined as a coefficient of plastic multiplier. As mentioned above, in the linear phase, i.e. no element yields, the plastic multiplier \( \{ \Delta \lambda \} = \{ 0 \} \). In order to solve the plastic multiplier after the element has yielded, we need to partition the relation \( \{ \Delta \Phi \} = [N]^T \{ \Delta Q^{el} \} - [A] \{ \Delta \lambda \} \) into active (a) and inactive (i) yield modes:

\[
\begin{bmatrix}
\{ \Delta \Phi_a \} \\
\{ \Delta \Phi_i \}
\end{bmatrix} =
\begin{bmatrix}
[N_a]^T \\
[N_i]^T
\end{bmatrix}
\begin{bmatrix}
\Delta Q^{el} \\
\Delta Q^i
\end{bmatrix} -
\begin{bmatrix}
[A_{aa}] & [A_{ai}] \\
[A_{ia}] & [A_{ii}]
\end{bmatrix}
\begin{bmatrix}
\{ \Delta \lambda_a \} \\
\{ \Delta \lambda_i \}
\end{bmatrix} -
\begin{bmatrix}
0
\end{bmatrix}
\]  

(32)

In the equation (32), the components of \( \{ \Delta \lambda_i \} \) corresponding to the inactive yield modes have been set to zero to comply with the constraint (28e). Then the above equation is decomposed according to the first block matrix:

\[
\{ \Delta \Phi_a \} = [N_a]^T \{ \Delta Q^{el} \} - [A_{aa}] \{ \Delta \lambda_a \}
\]  

(33)
We can assume that all active yield surfaces remain active, i.e., \( \{ \Delta \Phi_a \} = \{ 0 \} \), then the Hessian matrix \( [A_{aa}] \) is positive definite [5], and thus the plastic multiplier is:

\[
\{ \Delta \lambda_a \} = [A_{aa}]^{-1} [N_a] ^t \{ \Delta Q^{el} \} \\
\{ \Delta \lambda \} = \left\{ \begin{array}{c} \{ \Delta \lambda_a \} \\ \{ 0 \} \end{array} \right\} = \left[ [A_{aa}]^{-1} [N_a] ^t \{ \Delta Q^{el} \} \right] \\
\]

where \( \{ \Delta Q^{el} \} \) is computed by the equation (26). After obtaining the plastic multiplier, we can write the increment of displacement and the displacement of the step I at the nodes as follows:

\[
\{ \Delta u \} = \{ \Delta u^{el} \} + \{ \Delta u^p \} = \{ \Delta u^{el} \} + \left[ C_p \right] \{ \Delta \lambda \} \\
\{ u_I \} = \{ u_{I-1} \} + \{ \Delta u^p \} + \{ \Delta u^{el} \} 
\]

Substituting the nodal displacement into the subroutine of element response, we can obtain the element response stresses and deformations.

**EXAMPLES**

1. **The Nonlinear Dynamic Response Analysis of the Frame Structure Modeled by The Beam Element**

From the above analysis, we can see that the key of this new method is how to solve the increment of plastic deformation, i.e., the increment of plastic multiplier. The computation of plastic multiplier \( \{ \Delta \lambda_a \} \) is dependent on the Hessian matrix \( [A_{aa}] \), and the computation of \( [A_{aa}] \) is dependent on the hardening matrix, \( [H] \). It is shown in Appendix A that the computation of the hardening matrix is related to the stiffness of every stage. In general, the multi-linear-line hysteretic models consist of the incipient elastic stiffness, the strain hardening stiffness, the degrading unloading stiffnesses and the reloading stiffnesses, the last two are different from the incipient elastic stiffness and different from each other. If the unloading or reloading stiffness is regarded as an imaginary hardening stiffness, the hardening parameter in unloading or reloading stage can be obtained as shown in the Appendix A.

![Figure 4 Extended version of Taskeda’s model](image)
structures modeled by the multiple-vertical-line element model is basically the same, in this paper, the application of this new method is illustrated by a two story, one bay reinforced concrete frame structure [6] modeled by the beam element. The hysteretic model used in all the members is an extended version of Takeda’s model given by Powell etc. (Figure 4)[6]. The damping is assumed to be mass-dependent damping.

![Figure 5 Floor displacement envelope](image1)

![Figure 6 Positive moment envelope of left column](image2)

Figure 5 and 6 show the floor displacement envelope and the positive moment envelope of the left column (solid line), and the corresponding results computed by the program DRAIN-2D (dash line). We can see from the Figures that the results of this new method are in agreement with those of DRAIN-2D. This shows that the new method of nonlinear structural dynamic analysis is effective and reliable.

2 Nonlinear Dynamic Response Analysis for Hysteretic Model with Negative Stiffness

The problem of negative stiffness in nonlinear dynamic response analysis is very difficult, and is not overcome yet. The classical nonlinear response analysis methods can be divided into the iteration methods and the variable stiffness methods. The variable stiffness methods need to update and invert the stiffness matrix whenever the stiffness changes. When the negative stiffness appears, the stiffness matrix may be non-definite abnormal matrix. Therefore, at present, the iteration methods are usually used to deal with the nonlinear response problem with negative stiffness feature. But the iteration methods need a large number of repetitive iteration during the process of computation, so these methods waste time significantly. And when the negative stiffness appears, the common iteration methods usually can’t converge. Thus some special measures should be adopted, but these measures further increase computation time. The new method proposed can be successful in dealing with this problem.

It is known that the Hessain matrix $[A_{aa}]$ is the key in this method. Provided $[A_{aa}]$ is a positive definite matrix, the analysis can be implemented. It is proved in reference [5] that $[A_{aa}]$ is undoubtedly positive definitive under the circumstance of non-negative stiffness. Within the range of practical negative stiffness in the hysteretic behavior of common buildings, computation shows that $[A_{aa}]$ is positive definite as well. Owing to not having a large number of repetitive iterations or updating and inverting the stiffness matrix during the procedure of calculation, this method not only decreases quantities of computation and avoids probable unstable phenomenon as in the iteration
methods, but also avoids probable non-definite abnormal matrix as in the variable stiffness methods during dealing with the negative stiffness. For this reason, the computation time can be saved.

The example structure is a three-story shear-beam model frame structure. The interstory hysteretic model used in the analysis is shown in Figure 7. The structural parameters of 1~3 stories are: \( m = 507 \text{T}, 511 \text{T}, 338 \text{T}; \) \( K_0 = 4701 \text{KN/mm}, 4701 \text{KN/mm}, 4539 \text{KN/mm}; \) \( K_1 = 716 \text{KN/mm}, 716 \text{KN/mm}, 692 \text{KN/mm}; \) \( K_2 = -643 \text{KN/mm}, -643 \text{KN/mm}, -621 \text{KN/mm}; \) \( Q_c = 10521 \text{KN}, 9418 \text{KN}, 8126 \text{KN}; \) \( Q_u = 12319 \text{KN}, 11039 \text{KN}, 9515 \text{KN}. \) The input earthquake motion is the El-Centro waves. The peak acceleration is adjusted to 400gal, 620gal or 750gal. The structural collapse is taken as 82% ultimate strength after entering negative stiffness phase.

Figure 8~10 show the interstory displacement time history curves of the first story. We can see from these figures that at the ground peak acceleration of 400gal, the maximum interstory drift is 1/939. Only the first story enters the nonlinear state, but still doesn’t enter the negative stiffness stage. At the ground peak acceleration of 620gal, the maximum interstory drift gets to 1/495. Both the first and second stories enter nonlinear state, but only the first story enters the negative stiffness stage. At the ground peak acceleration of 750gal, the first interstory drift gets to 1/341, and the structure collapses at 4.9532 second. Figure 9 and 10 show that the zero lines of the time history curves shift upward. This is owing to the responses entering the negative stiffness stage in the positive direction, and is a rational and normal phenomenon.

Figure 11 and 12 show the interstory displacement-shear curves of the first story. We can see that these curves conform to the rules of the hysteretic model shown in Figure 7. The application example shows that this method is practical and effective to deal with the nonlinear dynamic response analysis of structures with negative stiffness restoring force behavior.
CONCLUSIONS

This paper extends the elastoplastic theory based on a pattern of self-equilibrating stresses to the nonlinear dynamic response analysis of structures. Its theoretical formulas are deduced and its procedure of calculation is described. Owing to introducing the concept of self-equilibrating stresses, which resists the stress redistribution caused by the deformation of the plastic hinge, every structural response is expressed as two components: the one that resists the increment of incipient external load, and the other one that resists the increment of the plastic deformation. Thus the traditional concept of revising the stiffness matrix when the stiffness changes in the classical methods is broken through.

The application examples show that this new method used for nonlinear dynamic response analysis of structures is effective and reliable, and specially has great advantage in dealing with the softening feature of negative stiffness. At the moment, the structural damping in analysis is assumed to be mass-dependant, and how to consider the stiffness-dependant damping need to research further.

A general computer program of nonlinear dynamic analysis based on this new method is in writing now. In order to bring the high efficiency of this new method into play, the optimum for the program is the task ahead.

REFERENCES


APPENDIX A

The hardening parameters $H$ and the strain hardening stiffnesses $k$ are both commonly used to describe a force-deformation relationship.

Consider the force-deformation relationship of Figure A-1(a). After a yield surface is activated, the constitutive laws can be expressed either in terms of the strain hardening stiffness $k_1$:

\[
\Delta M = k_1 \Delta \theta \tag{A-1}
\]

Fig. A-1  Force-deformation relationship

or in terms of the hardening parameter $H_1$ and the elastic stiffness $k_0$:

\[
\Delta M = H_1 \lambda \\
\Delta M = k_0 \Delta e \tag{A-2}
\]

The difference between the two relations is that in the first equation the parameter $\Delta \theta$ measures both the elastic and the plastic deformations, while in the second equation the elastic and the plastic deformations are measured separately by $\Delta e$ and $\lambda$, respectively. Hence, the deformation quantities $\Delta \theta$, $\lambda$, and $\Delta e$ are related through the following expressions:

\[
\Delta \theta = \lambda + \Delta e \tag{A-3}
\]

Substituting $\Delta \theta$, $\lambda$, and $\Delta e$ in the equations (A-1) and (A-2) into the equation (A-3), we can lead to an expression that relates the stiffness parameters $k_0$ and $k_1$ to the hardening parameter $H_1$: 
Similarly, for the relationship of Figure A-1(b), if we regard the unloading stiffness $k_u$ as an imaginary hardening stiffness, the hardening parameter $H_2$ can also be expressed by using the elastic stiffness $k_o$ and the imaginary hardening stiffness $k_u$:

$$\frac{1}{H_2} = \frac{1}{k_u} - \frac{1}{k_o}$$  \hfill (A-5)